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Geodesic Focusing and Space-Time Topology

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Given a space-time M and a point p in M, it is shown that, if the locus of first conjugate points of p along future-directed null geodesics consists of a single point, then M admits a compact (S^3) spacelike hypersurface. If in addition the null geodesics do not intersect before focusing, then, in a simply connected space-time, the spacelike hypersurface is a partial Cauchy surface.

1. INTRODUCTION

It is known that the existence of *conjugate points*[1] is a generic feature of space-times. Propositions 4.4.2 and 4.4.5 in Hawking and Ellis (1973) show that if physically reasonable energy conditions are satisfied then every causal geodesic which passes through a region with matter will either have a pair of conjugate points or be incomplete. The *conjugate locus* of a point p in a space-time M is defined as the set of points in M which are conjugate to p along some geodesic. In this paper we will be concerned with the structure of the conjugate locus and its implications on the topology and causal structure of the space-time. Specifically we consider the first null conjugate locus (FNCL) which is defined as the set of first[2] conjugate points to p along future-directed null geodesics. One could of course also define the FNCL for past-directed null geodesics.

The structure of the FNCL is also of great interest because of the connection between conjugate points and the global rotation of the universe discussed by the author in Rosquist (1980).

[1] If $\gamma(v)$ is a geodesic with tangent vector K^a then q is conjugate to p if there is a nontrivial solution to the geodesic deviation equations $D^2V^a/dv^2 = -R^a{}_{bcd}K^bV^cK^d$ which vanishes at both p and q.

[2] A first conjugate point (if any) always exists [see Rosquist (1982) for an explanation of this fact].

In the closed Friedmann models the FNCL consists of a single point. On the other hand, the FNCL in Gödel's universe contains a smooth closed nongeodesic null curve (Hawking and Ellis, 1973). The existence of such a curve implies that there are closed timelike lines in the model. In a general four-dimensional space-time the FNCL will be a two-dimensional surface with cusps. The classification of possible FNCLs in a space-time is an open problem. In this paper we deal exclusively with the case when all future-directed null geodesics through a point p converge to a single point, that point being the first conjugate point to p along all the null geodesics. This means that the FNCL of p reduces to a point. In that case it turns out that the space-time admits a compact spacelike hypersurface S which is topologically a 3-sphere. Furthermore, if the null geodesics do not intersect before the FNCL, then, if the space-time is simply connected, S is a partial Cauchy surface.[3] The purpose of this paper is to prove these statements. It is shown in Rosquist (1982) that the assumption that the FNCL is a single point is satisfied in the case of isotropic focusing.

2. PRELIMINARIES

We study a space-time M which is a four-dimensional connected C^∞ Hausdorff manifold with a C^∞ time-oriented Lorentz metric g. We use the notations and conventions of Hawking and Ellis (1973) unless otherwise stated.

Definition 2.1. A point q in M is an *absolute focusing point* of p in M if all future-directed null geodesics through p meet at q. If q is the first conjugate point to p along all null geodesics through p, then q is a *good* absolute focusing point of p.

The requirement in Definition 2.1. that a point has conjugate points in all null directions is a slightly stronger condition than to demand that any null geodesic has a pair of conjugate points.

The closed Friedmann models furnish an example of a class of space-times with good absolute focusing points while Gödel's model (Hawking and Ellis, 1973, p. 168) with one spacelike dimension suppressed exhibits absolute focusing points which are not good.

Let us introduce some notation. If $r \in M$ let $N_r^+ \subset T_r$ (or $N_r^- \subset T_r$) denote the set of future- (or past-) directed null vectors in T_r. Beside the space-time metric g we also use a natural Euclidean metric in T_r to define

[3]A partial Cauchy surface is a spacelike hypersurface which no causal curve intersects more than once.

the unit 3-sphere $S_r \subset T_r$. The Euclidean metric is defined by first choosing a Lorentz frame for g [i.e., $g_{ab} = \mathrm{diag}(-1, +1, +1, +1)$; $a, b = 0, 1, 2, 3$] and then change the sign of g_{00}. Further let P: $T_r - \{0\} \to S_r$ be the natural projection onto S_r and define $U_r^+ \equiv P(N_r^+)$, $U_r^- \equiv P(N_r^-)$. The set $U_r^+ \subset T_r$ represents the future-directed null geodesics through r. It is diffeomorphic to the 2-sphere.

The future null cone at p can be represented by a two-parameter smooth variation α: $I \times U_p^+ \to M$, $\alpha = \alpha(v, \xi)$, $v \in I$, $\xi \in U_p^+$, where $I = [0, 1]$ and v is an affine parameter along the null geodesics. Then for an arbitrary fixed ξ, $v \to \alpha(v, \xi)$ is a null geodesic with the affine parameter v chosen such that $\alpha(0, \xi) = p$. We let $\alpha_*(w, \xi)$ denote the tangent vector of the null geodesic $v \to \alpha(v, \xi)$ at the point $\alpha(w, \xi)$. Note that $\alpha_*(0, \xi) \in T_p$ is *not* independent of ξ. We may express α by means of the exponential map as $\alpha(v, \xi) = \exp_p[v\alpha_*(0, \xi)]$.

3. THE NULL GEODESIC SPHERE

We assume throughout that q is a good absolute focusing point of p. Let N be the null geodesic surface between p and q. Thus $r \in N$ iff there is a null geodesic segment γ: $I \to M$ with $\gamma(0) = p, \gamma(1) = q, \gamma(v) \neq q$ when $v \in (0, 1)$, and a $v \in I$ such that $\gamma(v) = r$. The map α is a smooth immersion when $v \neq 0$ and $\alpha(v, \xi)$ is not a conjugate point of p along $v \to \alpha(v, \xi)$. Therefore, intuitively, it seems that N is an immersion of the 3-sphere which is smooth except at p and q where it is only continuous. This will be verified in the following lemmas. The first lemma shows that the affine parameters can be chosen such that the null geodesics correspond to longitudinal lines between the "poles" p and q while the latitudinal surfaces are given by $v = \text{const}$, where v is the affine parameter.

Lemma 3.1. The affine parameters of the null geodesics through p can be chosen such that $\alpha(0, \xi) = p$ and $\alpha(1, \xi) = q$ for all ξ.

Proof. Since α is assumed to be a smooth variation it is sufficient to show that the function f: $U_p^+ \to [0, \infty)$, which to a given null geodesic assigns the affine parameter value at q, is differentiable because then one can use f to rescale the affine parameters smoothly by the transformation $v \to v/f$.

We will use the implicit function theorem in the form given in Dieudonné (1960), (10.2.2). Choose a normal neighborhood U with origin at q and let φ: $U \to R$ be a normal time coordinate on U. Next choose $v_0 \in \mathring{I}$ and $\xi_0 \in U_p^+$ such that $\alpha(v_0, \xi_0) = q$. ($\mathring{I}$ denotes the interior of I. One can always make a uniform rescaling of v to make q belong to the range of α, i.e., $v \to Av$ where A is a constant on U_p^+.) Define G: $I \times U_p^+ \to R$ by[4] $G \equiv \varphi \circ \alpha$. Then $G(v_0, \xi_0) = 0$ and

$$(\partial G/\partial v)|(v_0, \xi_0) \neq 0 \tag{1}$$

since φ is a strictly increasing function along a null geodesic. Then by the implicit function theorem there is a neighborhood W of ξ_0 and a smooth function $\bar{f}$: $W \to R$ with $G(\bar{f}(\xi), \xi) = 0$. The function $\bar{f}$ can be identified with the function f defined above. Thus f is smooth. ■

From now on we assume that the affine parameters are chosen such that $\alpha(0, \xi) = p$ and $\alpha(1, \xi) = q$. If n and s are the "poles" of S^3, then $S^3 - \{n, s\}$ and $\mathring{I} \times U_p^+$ are diffeomorphic. Thus we can extend $\alpha|\mathring{I} \times U_p^+$ to a map h: $S^3 \to M$ by defining

$$h|\mathring{I} \times U_p^+ \equiv \alpha|\mathring{I} \times U_p^+ \tag{2}$$

and

$$h(s) \equiv p, \qquad h(n) \equiv q \tag{3}$$

Then $N = h(S^3)$.

Remark 3.2. The map h is locally one-to-one at n. For let U be a normal neighborhood of q. Then if v_1 and v_2 are sufficiently near 1, $h(v_1, \xi_1)$ and $h(v_2, \xi_2)$ lie on the past light cone of q in U. Hence if $h(v_1, \xi_1) = h(v_2, \xi_2) \neq q$ then $\xi_1 = \xi_2$ since geodesics cannot bifurcate. But then also $v_1 = v_2$.

Proposition 3.3. The map h is an immersion of the 3-sphere in M. Further h is differentiable except at p and q where it is continuous.

Proof. We need only observe that h, being essentially the exponential map, is a local diffeomorphism (LD) except at s and at points corresponding to conjugate points to p. Then since q is the first conjugate point to p (along null geodesics), h is a LD except at s and n. The continuity of h and h^{-1} at s and n follows directly from the definition and Remark 3.2. ■

Next we show that the variation α which represents the light cone at p also represents the light cone at q. This seems again rather obvious but nevertheless we wish to give a formal motivation. For that purpose let U be a normal neighborhood of q and let $v_0 \in \mathring{I}$ be a fixed number such that $\alpha(v_0, \xi) \in U$ for all ξ. We define a function F: $U_p^+ \to U_q^-$ by $F \equiv P \circ \exp_q^{-1} \circ \alpha_0$, where α_0: $U_p^+ \to M$ is the function defined by $\xi \to \alpha(v_0, \xi)$.

[4] Strictly, G is only defined on $\alpha^{-1}(U)$.

Proposition 3.4. The variation α which represents the future light cone of p also represents the past light cone of q.

Proof. We show that F is a diffeomorphism onto U_q^-. First we observe that if F is a diffeomorphism then it must necessarily be onto. This is because any homeomorphism of a connected space into itself is onto. We further note that F is one-to-one by Remark 3.2. What is left to show is that F is a LD. But α_0 is a LD of U_p^+ onto its image in M considered as a submanifold with the relative topology and similarly $\exp_q^{-1}$ and P suitably restricted are also LDs. Thus F is a LD. ■

The following lemma is useful when discussing self-intersections of α. It will not be used in this paper.

> *Lemma 3.4.* If M is causal, i.e., contains no closed causal curves, and if q is a good absolute focusing point of p, then there are normal neighborhoods U of p and V of q such that $N \cap U = E^+(p, U)$ and $N \cap V = E^-(q, V)$ where $E^+(E^-)$ is the horismos (here it is just the local light cone).

Proof. First we observe that the causality conditions imply that $p \neq \alpha(v, \xi)$ for $v \in (0,1]$ and $\xi \in S^2$. Choose $\epsilon, 0 < \epsilon < 1$, such that the set $B = \{\alpha(v, \xi) | 0 \leqslant v < \epsilon, \xi \in S^2\}$ is contained in a normal neighborhood W of p. Then $N - B$ is a closed set and $p \notin N - B$. Thus there is a convex normal neighborhood $U \subset W$ of p such that $U \cap (N - B) = \varnothing$. Then $N \cap U = B \cap U = E^+(p, U)$, which shows that U is the desired neighborhood of p. An analogous argument for q completes the proof. ■

4. THE PARTIAL CAUCHY SURFACE

The following proposition shows that the null geodesic sphere can be made spacelike everywhere.

Proposition 4.1. The null geodesic sphere can be deformed to a smooth immersed spacelike S^3 surface.

Proof. Let the null geodesic sphere be represented by the variation $\alpha(v, \xi) = \alpha$: $I \times U_p^+ \to M$. First we deform N at p and q to obtain a smooth surface. To that end remove the "caps", $\{\alpha(v, \xi) | 0 \leqslant v < \delta\}$ and $\{\alpha(v, \xi) | 1 - \delta < v \leqslant 1\}$, where δ is chosen so that the caps are contained in convex normal neighborhoods of p and q, respectively. Let N' denote N with the caps removed, i.e., $N' = \{\alpha(v, \xi) | \delta \leqslant v \leqslant 1 - \delta\}$. Now clearly we may extend N' to a set $\tilde{N}$ by attaching new smooth spacelike caps which join smoothly to N'. Then $\tilde{N}$ is a smooth immersion of S^3.

Next we observe that we can define a smooth future-directed timelike vector field V along N' [i.e., an assignment $(v, \xi) \to V$ where $V \in T_{\alpha(v,\xi)}$] such that $g(V, V) = g(V, K) = -1$ where K is the tangent vector along the null geodesics. Further V can be extended to a smooth future-directed timelike vector field along $\tilde{N}$. Now suppose W is a subset of S^3 which corresponds to $[\delta, 1-\delta] \times U_p^+$ and let ϕ: $S^3 \to \tilde{N}$ be an immersion which coincides with the variation α on W. Define a deformation of N' by a function ψ_ϵ: $W \to M$,

$$\psi_\epsilon(\rho) \equiv \exp_{\phi(\rho)}(-v\epsilon V_\rho) \tag{4}$$

where $\rho = (v, \xi) \in W$, $V_\rho \in T_{\phi(\rho)}$, and $\epsilon > 0$ is a parameter. Further, there is a smooth real positive function on S^3 which is equal to v on W. Hence ψ_ϵ may be extended to a smooth function on S^3 which can be regarded as a deformation $\tilde{N}$ which sends every point of $\tilde{N}$ to the past by an amount $v\epsilon$ along the geodesic whose tangent at $\tilde{N}$ is V.

We have to show that the surface $S \equiv \psi_\epsilon(S^3)$ is everywhere smooth and spacelike when ϵ is sufficiently small. The idea is to use local tangent frames (R_1, R_2, R_3) on $\tilde{N}$, where $R_1 = K$ on N', and show that for small ϵ the images of R_1, R_2, and R_3 under $\psi_{\epsilon*}$ are spacelike. (Strictly speaking we should use a tangent frame on S^3 since $\psi_{\epsilon*}$ maps tangent vectors on S^3 to tangent vectors on M. However, in order not to become bogged down in technicalities we do not bother about this.) We define

$$R_\epsilon \equiv \psi_{\epsilon*}(\partial/\partial v) \tag{5}$$

If $s \in N'$ the coordinates (v, ξ) on N' can be naturally extended to coordinates (t, v, ξ) on a neighborhood U of s using V and the exponential map. Then if $t = 0$ on N' we have on U

$$V \equiv \partial/\partial t, \qquad K \equiv \partial/\partial v \tag{6}$$

$$[V, K] = 0 \quad \text{or} \quad \nabla_V K = \nabla_K V \tag{7}$$

$$g(V, V) = g(V, K) = -1 \tag{8}$$

Further from (4) and (5) we see that

$$R_\epsilon = K - \epsilon V \tag{9}$$

$$g(R_\epsilon, R_\epsilon) = g(K, K) + 2\epsilon - \epsilon^2 \tag{10}$$

Expanding $g(K, K)$ in $t = -v\epsilon$ we obtain using (7) and (8),

$$g(K, K) = (C/2)v^2\epsilon^2 \tag{11}$$

where C is the value of $V[V(g(K, K))]$ for some t_1 in $[-v\epsilon, 0]$.

Now fix $\epsilon_0 > 0$ and let $\epsilon < \epsilon_0$. Since $V[V(g(K, K))]$ is finite when (t, v, ξ) runs over the compact set $[-\epsilon_0, 0] \times I \times U_p^+$, it follows from (10) that R_ϵ is spacelike for all $(v, \xi) \in I \times U_p^+$ when ϵ is sufficiently small. Again appealing to the compactness of $[-\epsilon_0, 0] \times I \times U_p^+$ we realize that R_2 and R_3 stay spacelike for all (v, ξ) when mapped by $\psi_{\epsilon *}$ if ϵ is small. Hence for small ϵ, $\psi_\epsilon([\delta, 1-\delta] \times U_p^+)$ is spacelike. Further since the caps have compact closure the images of R_1, R_2, and R_3 under $\psi_{\epsilon *}$ are spacelike everywhere on the deformed caps if ϵ is small. Thus for small ϵ, the whole surface $\psi_\epsilon(S^3)$ is spacelike. Finally, we note that by the compactness of S^3 the surface $\psi_\epsilon(S^3)$ is everywhere smooth for sufficiently small ϵ. This is because the geodesic congruence defined by V near a point $r \in \tilde{N}$ has no caustics in a normal neighborhood of r. ■

Now we make the simplifying assumption that h is a one-to-one immersion, or in other words that the null geodesics do not intersect except when $v = 0$ or 1. Then since S^3 is compact, h is an embedding (cf. Hawking and Ellis, 1973, p. 23).

If M is simply connected we may use the following topological theorem (Hirsch, 1976, Theorem 4.6):

> *Theorem.* If M is a simply connected manifold and $N \subset M$ is an embedded connected closed compact hypersurface, then N separates M.

A set A is said to be separated by B if $A - B$ is the disjoint union of two open sets. Then we have the following proposition:

Proposition 4.2. If M is simply connected and the null geodesic sphere N has no self-intersections, then the space-time is separated by the sphere.

We proceed to characterize the components of $M - N$. For that purpose we define the following:

Definition 4.3. $F_N \equiv \{r \in M \mid$. There is a smooth curve $\gamma(v)$ from $s \in N$ to r such that $\gamma_*|s$ is future directed and $(s, r]_\gamma \cap N = \varnothing\}$.

Note. If $\gamma(a) = s$ and $\gamma(b) = r$ then $(s, r]_\gamma \equiv \{\gamma(v) | 0 < v \leq 1\}$. The set P_N is given by the time-reversed definition. By the definition we have $F_N \cap N = P_N \cap N = \varnothing$.

Lemma 4.4. The connected components of $M - N$ are F_N and P_N.

Proof. F_N is open since if $r \in F_N$ then since N is closed there is a convex normal neighborhood U of r with $U \cap N = \varnothing$. Then any point in U can be joined to r by a smooth curve lying entirely in U so that $U \subset F_N$. Similiarly P_N is open. Next we observe that since M is connected it is also arc-connected. Further, any arc can be approximated by a smooth curve. Hence any point of M can be joined to N by a smooth curve. Thus $F_N \cup P_N \cup N = M$. The sets F_N and P_N are arc-connected and hence connected subsets of M. To see that, let r_1 and r_2 be points in F_N and $\gamma_1(v)$ and $\gamma_2(v)$ curves with $\gamma_i(0) = s_i \in N$, $\gamma_i(1) = r_i$ and $\gamma_i(v) \in F_N$ for $v \in (0, 1]$ and $i = 1, 2$. Let σ be a curve in N joining s_1 and s_2. The curve which is the sum of $-\gamma_1$, σ, and γ_2 can be deformed by pushing σ slightly to the future into F_N so that the resulting curve lies in F_N and connects r_1 and r_2. Thus F_N is connected. Analogously P_N is connected. Since $M - N$ has at least two components by Theorem 4.2, F_N and P_N are disjoint. ■

Corollary 4.5. The null geodesic sphere N is achronal, that is, no two points of N can be joined by a timelike curve.

Note that if h is an embedding, the deformed sphere S is also an embedding if ϵ is small. Hence Proposition 4.2, Lemma 4.4, and Corollary 4.5 all apply to the spacelike sphere S as well.

Finally, we can now state our main theorem.

Theorem 4.6. If M is a space-time and if there is a $q \in M$ which is a good absolute focusing point of $p \in M$, then the space-time admits a compact spacelike S^3 hypersurface S. If in addition the null geodesics through p do not intersect before q and M is simply connected, then S is a partial Cauchy surface.

5. PROSPECTS

Although we have stated our results in terms of four-dimensional space-times, the same arguments can be used when $\dim(M) \geqslant 3$.

In Theorem 4.6 it is assumed that all future-directed null geodesics from a point p reconverge to a single point q. This implies a high degree of symmetry which the real universe does not possess. However, one would expect that the convergence condition could be relaxed so that the FNCL need not be a single point, but need only be contained in a suitably small set. There is another condition in Theorem 4.6 which may possibly be weakened, namely, the assumption that the null geodesics do not intersect before q.

Finally we think that it would be of great interest to obtain results on the space-time structure when the FNCL is not contained in a normal neighborhood. In particular one would like to know under what conditions the FNCL contains a closed null curve (cf. Rosquist, 1980).

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